INTEGRAL EQUATIONS OF CONVOLUTION OF THE FIRST KIND ON A SYSTEM OF SEGMENTS OCCURRING IN THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

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The integral equation of convolution given on an arbitrary number of segments for quite general kernels is examined. This equation contains the integral equations of contact problems for a layer which is at rest on a rigid linearly deformable base. It also contains the integral equations of problems in the creep theory and the integral equations of some classic dynamic contact problems. Many mixed problems in the theory of elasticity, hydromechanics, and mathematical physics are reduced to Equation (1.1) (review [1]).

The Equation (1.1) is solved in closed form [2] only in very special cases. In most cases it must be solved by one or another approximate method. In almost all efforts in this area Equation (1.1) is analyzed under the assumption that $N = 1$. One of the approximate methods of solution is based on the utilization of approximate "factorization" which is a special representation of some functions describing the integral equation [3, 4]. In this case the effectiveness of the obtained approximate solution is then checked on particular examples. Papers in which the method of approximate factorization is justified, i. e. in which estimates are given for the accuracy of the approximate solution, are not known to the author.

In this paper the solvability of the equation in spaces which are important for applications, and the classes of correctness are established. The method of approximate factorization is substantiated. At the same time a method is given which allows to construct for Equation (1.1) an approximate solution which in a uniform metric differs arbitrarily little from the exact solution.

1. The integral equation of the following form is analyzed:

$$\mathbf{K}q \equiv \sum_{k=1}^{N} \int_{a_{2k-1}}^{a_{2k}} k(x-\xi)\, q_{2k-1}(\xi)\, d\xi = 2\pi f_{2m-1}(x) \equiv 2\pi f(x) \qquad (1.1)$$

$$a_{2m-1} \leqslant x \leqslant a_{2m}, \quad |a_1| < \infty, \quad |a_{2N}| < \infty \quad (m = 1, 2, \ldots, N)$$

the kernel $k(t)$ can be represented in the form

$$k(t) = \int_{-\infty}^{\infty} K(u)\, e^{iut} du \qquad (1.2)$$

We will assume that

1) the function $K(u)$ is continuous, real and even on the x-axis,
2) the function
$$K(u) > 0, \quad |u| < \infty$$
3) the function
$$K(u) = c^2 u^{-2\gamma}\,[1 + O(u^{-\delta})], \quad u \to \infty, \quad 0 < \gamma < 1 \qquad (1.3)$$

Here δ satisfies the following inequalities

$$\delta > \gamma \quad \text{for} \quad \gamma \geqslant 0.5, \qquad \delta > 1-\gamma \quad \text{for} \quad \gamma < 0.5$$

Let us introduce a number of definitions for spaces needed later.

1.1°. Let us designate through H_γ the set of the following functions $q(x)$:

$$\|q\|_H^2 = \int_{-\infty}^{\infty} K(u)\,|Q(u)|^2\,du < \infty, \quad Q(u) = \int_{-\infty}^{\infty} q(x)\,e^{iux}dx \tag{1.4}$$

Evidently, elements from H_γ belong to some range of Hilbert spaces [5].

1.2°. Through $S(\sigma)$ and $s(\sigma)$, respectively, we designate spaces of complex sequences $X = \{x_n\}$, which converge with the weight n^σ. In the case of $s(\sigma)$ here the convergence is to zero, i.e.

$$\lim |n^\sigma x_n| = c \quad (n \to \infty), \quad \sigma \geqslant 0, \quad c < \infty$$

In each of the spaces we introduce a norm through the relationship

$$\|X\| = \sup_n |n^\sigma x_n|$$

1.3°. We denote by $C_k^\lambda(a, b)$ the space of functions for which the kth order derivative on $[a, b]$ satisfies Hoelder's condition with an index $0 < \lambda < 1$ and a norm

$$\|f\|_{C_k^\lambda(a,b)} = \sum_{n=0}^{k} \max_x |f^{(k)}(x)| + \max_{x,y} |f^{(k)}(x) - f^{(k)}(y)|\,|x-y|^{-\lambda}$$
$$x, y \in [a, b]$$

For $k = \lambda = 0$ we have that $C(a, b)$ is the space of continuous on $[a, b]$ functions.

1.4°. $L_p(a, b)$ is the space of absolutely summable on $[a, b]$ functions of a degree $p \geqslant 1$ and an ordinary norm.

1.5°. We will say that $f(x) \in E$ if for its Fourier transform $F(\lambda)$ the following relationship holds:

$$\|f\|_E = \int_{-\infty}^{\infty} \left|\frac{F(\lambda)}{K(\lambda)}\right| d\lambda < \infty, \quad f(x) = \int_{-\infty}^{\infty} F(\lambda)\,e^{i\lambda x}d\lambda \tag{1.5}$$

The value of the function $f(x)$ on the segment $[a_{2k-1}, a_{2k}]$ will be designated by $f_{2k-1}(x)$. It is apparent that $f(x) \in C(-\infty, \infty)$.

1.6°. By $C(\gamma)$ we will designate the set of functions which are continuous with the weight $(x - a_{2k-1})^\gamma (a_{2k} - x)^\gamma$ on segments $[a_{2k-1}, a_{2k}]$.

The function $f(x) \in C(\gamma)$, if

$$\|f\|_{C(\gamma)} = \sup_k \max_x |(x - a_{2k-1})^\gamma (a_{2k} - x)^\gamma f(x)| < \infty$$
$$x \in [a_{2k-1}, a_{2k}]$$

2. 1°. Equation (1.1) is the usual equation of convolution in which the unknown function $q(x)$ becomes zero outside the segments $[a_{2k-1}, a_{2k}]$, $k = 1, 2, \ldots, N$. In contact problems of the theory of elasticity [1] the values of the function $q(x)$ on segments $[a_{2k-1}, a_{2k}]$ represent the unknown contact stresses $q_{2k-1}(x)$ under the stamp. On this segment the values of the function $f(x)$ are known. These values for $x \in [a_{2k-1}, a_{2k}]$ will be designated by $f_{2k-1}(x)$. The function $f(x)$ characterizes the displacements of points on the surface of the layer. Outside the segments $[a_{2k-1}, a_{2k}]$ the values of the function $f(x)$ are unknown. The function $f_{2k-1}(x)$ characterizes the shape of the foundation of the stamp and the depth of penetration of the stamp into the layer.

The preliminaries described here are helpful for the selection of the necessary classes of functions which must be used for the analysis of Eq. (1.1).

The class of functions $q(x)$ must be shch that:

(1) the function $f(x)$ must be bounded for $|x| < \infty$;

(2) the energy which is accumulated by the elastic body when stamps are pressed into it, must be finite for stamps acting on finite segments.

2°. Let us study the properties of the function $k(t)$. Using known theorems from the theory of Fourier integrals [6], we obtain the following result.

Lemma 2.1. The following estimates are valid for $t \to 0$:

$$k(t) = O(t^{2\gamma-1}), \quad \gamma < 0.5; \qquad k(t) = O(\ln|t|), \quad \gamma = 0.5$$

$$k(t) = O(1), \quad \gamma > 0.5$$

For $|t| > \varepsilon > 0$ the function $k(t)$ is continuous.

On the basis of Lemma 2.1 the following theorem is proved.

Theorem 2.1. The operator K acts continuously from L_p into $C(-T, \ T)$ Here $(2\gamma)^{-1} < p < \infty$, $\gamma \leqslant 0.5$; $1 < p < \infty$, $0.5 < \gamma$, $T < \infty$.

Under conditions of Theorem 2.1 apparently requirement (1) of the previous section is satisfied. The condition which permits to assert the fulfilment of property (2) of the previous point gives

Lemma 2.2. Any space L_p, $(\gamma + 0.5)^{-1} < p < \infty$, $\gamma \leqslant 0.5$ and $1 < p < \infty$, $0.5 < \gamma$ is imbedded in H_γ.

The proof of the Lemma follows from the boundedness of the operator of the Fourier transformation [7] acting from L_p, $1 < p \leqslant 2$ into L_q, $q = p(p-1)^{-1}$.

Theorem 2.2. In the space L_p $\{p = 2, \gamma \leqslant 0.25; \ (2\gamma)^{-1} < p \leqslant 2, \ 0.25 < \gamma \leqslant 0.5; \ 1 < p \leqslant 2, \ 0.5 < \gamma\}$ the equation (1.1) cannot have more than one solution.

Proof. Multiplying Eq. (1.1) by $q(x) \in L_p$ and integrating along the entire axis, we obtain

$$\|q\|_H^2 = \sum_{k=1}^{N} \int_{a_{2k-1}}^{a_{2k}} f_{2k-1}(x)\, q_{2k-1}(x)\, dx \tag{2.1}$$

By virtue of Theorem 2.1 and Lemma 2.2 the relationship (2.1) is correct. It follows from (2.1) that if $f_{2k-1}(x) \equiv 0$, $x \in [a_{2k-1}, a_{2k}]$, then $q(x) \equiv 0$, $|x| < \infty$. The theorem is proved.

3°. Theorem 2.2 is valid for a more general equation.

Let $M(u)$ be a real function, even and continuous on the real axis, which has the property

$$M(u) = O(u^{-2\gamma-\delta}), \qquad u \to \infty \tag{2.2}$$

The value of δ is given in Sect. 1.

Let us examine the perturbed equation (1.1) of the form

$$\mathbf{K}q + \lambda \mathbf{M}q = 2\pi f, \quad \mathbf{M}q \equiv \sum_{k=1}^{N} \int_{a_{2k-1}}^{a_{2k}} m(x-\xi)\, q_{2k-1}(\xi)\, d\xi \tag{2.3}$$

Here λ is a complex parameter. The function $m(x)$ has the form

$$m(x) = \int_{-\infty}^{\infty} M(u)\, e^{iux} du \qquad (2.4)$$

Theorem 2.3. Let

$$\|MK^{-1}\|_{C(-\infty,\infty)} = \max |M(u)\, K^{-1}(u)| = \varkappa < 1 \quad (|u| \leqslant \infty) \qquad (2.5)$$

Then for Eq. (2.3) Theorem 2.2 is valid if λ lies within the circle

$$|\lambda| < \varkappa^{-1} \qquad (2.6)$$

The proof of Theorem 2.3 is analogous to the proof of Theorem 2.2 if in Eq. (2.3) the real and imaginary parts are separated and examined as separate equations.

Theorems 2.2 and 2.3 prove the uniqueness for Eqs. (1.1) and (2.3). The following Sections are devoted to the proof of existence of a solution.

3. The existence of a solution for Eq. (1.1) is established with the aid of the well known method of perturbations. For this purpose the operator $\mathbf{K}$ is split in two in a special manner : $\mathbf{K}_s$ and $\mathbf{M}_s$. The first one of these will turn out to be invertible, the second one small in some space. After this the proof of existence of a solution will not present any difficulties.

1°. The process of decomposition is started with the construction of a special representation of the function $K(u)$.

Lemma 3.1. The following representation is valid:

$$K(u) = K_s(u) + M_s(u), \qquad -\infty \leqslant u \leqslant \infty \qquad (3.1)$$

The function $K_s(u)$ satisfies conditions (1)–(3) in Sect. 1, it is meromorphic, and in the complex plane it has single poles ζ_n and single zeros z_n with a sufficiently large modulus. The first s zeros can have a finite multiplicity greater than one.

Taking into account multiplicity, the zeros z_n and poles ζ_n of the upper half-plane have the asymptotics

$$z_n \sim i(\beta n + b) \quad \beta, b, g > 0; \qquad \zeta_n \sim i(\beta n + g) \quad n \to \infty \qquad (3.2)$$

The function $M_s(u)$ has the property (2.2), and in addition

$$\|M_s\|_{C(-\infty,\infty)} = \max_u |M_s(u)| \to 0, \qquad s \to \infty \qquad (3.3)$$

The representation (3.1) can be obtained on the basis of a theorem of completeness [8], which in this paper is utilized in the form of the following lemma.

Lemma 3.1. Let the even, real, and continuous, on the entire real axis, function $\varphi(u)$ vanish at infinity. Then in $C(-\infty, \infty)$ this function can be approximated by the following functions:

$$\varphi_k(u) = (u^2 + \lambda_k^2)^{-1}, \qquad \lambda_k = \sigma k + \tau > 0$$

Let us apply Lemma 3.1 to the function

$$\varphi(u) = \frac{K(u) - K_0(u)}{K_0(u)}, \quad K_0(u) = \frac{\Gamma(g/\beta + iu/\beta)\,\Gamma(g/\beta - iu/\beta)}{\Gamma(b/\beta + iu/\beta)\,\Gamma(b/\beta - iu/\beta)} \qquad (3.4)$$

$$b - g = \gamma\beta, \qquad c = \beta^\gamma$$

Here $\Gamma(z)$ is Euler's gamma function.

Selecting $i\lambda_k$ to be different from the poles of the function $K_0(u)$, we obtain

$$\varphi(u) = \sum_{k=1}^{s} c_k \varphi_k(u) + R_s(u), \quad R_s(u) = O(u^{-\delta}), \quad u \to \infty \tag{3.5}$$

Here $R_s(u)$ is the residual term of the approximation, and c_k are the coefficients of approximation.

Since $\| R_s(u) \|_{c(-\infty,\infty)} \to 0$ for $s \to \infty$, it is possible to find such an s_0 that for $s > s_0$ the representation (3.1) will be valid with all indicated properties. In this connection

$$K_s(u) = K_0(u)\left[1 + \sum_{k=1}^{s} c_k\varphi_k(u)\right], \quad M_s(u) = K_0(u)R_s(u) \tag{3.6}$$

2°. Now let us take up the analysis of Eq. (1.1) with the kernel (1.2) in which the role of the function $K(u)$ is played by $K_s(u)$.

Equations (1.1) with meromorphic function $K_s(u)$ were already studied for $N = 1$ in papers [9–11].

The integral equations are reduced to an infinite system of linear equations which turns out to be uniquely solvable [11]. The solvability of the system can be proven because of the minimality properties of exponential functions on a finite segment established in [12] (p. 133) and in [13].

Other methods also exist for the reduction of the integral equation or the corresponding boundary value problem [14, 15] to an infinite system. In this connection the obtained infinite systems are identical to the ones in [9–11, 16], however, the method used does not permit the proof of their solability as a whole.

We will seek a solution of the integral equation (1.1) with the right side (1.5) in the form of a series

$$q_{2k-1}(x) = \int_{-\infty}^{\infty} \frac{F(\eta)}{K_s(\eta)} e^{i\eta x} d\eta + \sum_{l=1}^{\infty}\sum_{v=0}^{p(l)} [x_l(2k-1, v)(x - a_{2k-1})^v \exp iz_l(x - a_{2k-1}) +$$
$$+ y_l(2k-1, v)(a_{2k} - x)^v \exp iz_l(a_{2k} - x)]\, i^v \tag{3.7}$$
$$a_{2k-1} \leqslant x \leqslant a_{2k}$$

Here $p(l) + 1$ is the multiplicity of the zero z_l, which lies on the upper half-plane. It is assumed that the numbering of zeros z_l and poles ζ_l, of the upper half-plane is carried out in the order of increasing moduli and arguments (in the case of equal moduli).

Representing the kernel $k(t)$ in the form of a series in residues and utilizing (3.7), we arrive after integration at an infinite system for

$$X_k = \{x_l(2k-1, v)\}, \qquad Y_k = \{y_l(2k-1, v)\}$$

$$AX_k + C_kY_k + \sum_{m=1}^{k-1} [B(1, m)X_m + B(2, m)Y_m] = L_k$$

$$AY_k + C_kX_k + \sum_{m=k+1}^{N} [D(1, m)Y_m + D(2, m)X_m] = G_k \tag{3.8}$$

$$X_0 = Y_0 = X_{N+1} = Y_{N+1} = 0, \qquad k = 1, 2, \ldots, N$$

$$AX_k = \left\{\sum_{l=1}^{\infty}\sum_{v=0}^{p(l)} a_{r,l}(v)\, x_l(2k-1, v)\right\}$$

starting with some l, all $p(l) \equiv 0$.

Here

$$A = \{a_{r,l}(v)\} = \{D^v(\zeta_r - z_l)^{-1}\}, \quad D^v f(z_l) = d^v f(z_l)/dz_l^v$$

$$C_k = \{c_{r,l}(v)\} = \{D^v(\zeta_r + z_l)^{-1} \exp iz_l (a_{2k} - a_{2k-1})\}$$

$$B(1, m) = \{b_{r,l}(1, v)\} = \{D^v(\zeta_r - z_l)^{-1} \langle \exp i\zeta_r (a_{2k-1} - a_{2m-1}) - \exp i[z_l(a_{2m} - a_{2m-1}) + \zeta_r(a_{2k-1} - a_{2m})]\rangle\}$$

$$B(2, m) = \{b_{r,l}(2, v)\} = \{D^v(\zeta_r + z_l)^{-1} \langle \exp i[z_l(a_{2m} - a_{m-1}) + \zeta_r(a_{2k-1} - a_{2m-1})] - \exp i\zeta_r(a_{2k-1} - a_{2m})\rangle\}$$

$$D(1, m) = \{d_{r,l}(1, v)\} = \{D^v(\zeta_r - z_l)^{-1} \langle \exp i\zeta_r(a_{2m} - a_{2k}) - \exp i[z_l(a_{2m} - a_{2m-1}) + \zeta_r(a_{2m-1} - a_{2k})]\rangle\}$$

$$D(2, m) = \{d_{r,l}(2, v)\} = \{D^v(\zeta_r + z_l)^{-1} \langle \exp i[z_l(a_{2m} - a_{2m-1}) + \zeta_r(a_{2m} - a_{2k})] - \exp i\zeta_r(a_{2m-1} - a_{2k})\rangle\}$$

$$L_k = \{l_r(k)\} = \left\{\sum_{m=1}^{k-1} t_r(m) - e_r(k)\right\}, \qquad G_k = \{g_r(k)\} = \left\{\sum_{m=k+1}^{N} \tau_r(m) - s_r(k)\right\}$$

$$e_r(k) = \int_{-\infty}^{\infty} \frac{F(\eta) \exp i\eta a_{2k-1}}{(\zeta_r - \eta) K_s(\eta)} d\eta, \quad s_r(k) = \int_{-\infty}^{\infty} \frac{F(\eta) \exp i\eta a_{2k}}{(\zeta_r + \eta) K_s(\eta)} d\eta$$

$$t_r(m) = \int_{-\infty}^{\infty} \frac{F(\eta)}{K_s(\eta)(\zeta_r - \eta)} \{\exp i[\zeta_r(a_{2k-1} - a_{2m}) + \eta a_{2m}] - \exp i[\zeta_r(a_{2k-1} - a_{2m-1}) + \eta a_{2m-1}]\} d\eta$$

$$\tau_r(m) = \int_{-\infty}^{\infty} \frac{F(\eta)}{K_s(\eta)(\zeta_r + \eta)} \{\exp i[\zeta_r(a_{2m-1} - a_{2k}) + \eta a_{2m-1}] - \exp i[\zeta_r(a_{2m} - a_{2k}) + \eta a_{2m}]\} d\eta$$

For the transition from the integral equation to the infinite system to be legitimate, it is necessary [11–13] to establish the possibility of representing the function $K_s(u)$ in the form of a ratio of two entire functions $P(iu)$ and $Q(iu)$. The indicatrices of growth of these functions must be equal to $\sigma|\sin\varphi|$, $\sigma > 0$.

For entire functions representing $K_s(u)$ according to (3.6) with the asymptotic of zeros (3.2) the mentioned properties of indicatrices of growth were established in [17] (p. 144).

The properties of operators generated by matrices A, B, C, D are examined in [9–11, 18].

The following lemma is necessary for later application.

Lemma 3.2. The operator $A^{-1}R$ is continuous from any $S(\sigma)$, $\sigma > 0$ into $S(1-\gamma)$. Here R is any of the operators C, B, D. The elements of the matrix A^{-1}, which is the bilateral inverse of A, are given by the relationship

$$A^{-1} = \{\tau_{l,r}(v)\} = \{H_r(-z_l, v)[K_-^{-1}(\zeta_r)]'^{-1}\}$$

$$H_r(-z_l, v) = \frac{(-1)^v}{p(l)!} \left[\frac{(\alpha + z_l)^{p(l)+1}}{K_+(\alpha)(\alpha + \zeta_r)}\right]^{(p(l)-v)} C_{p(l)}^{\,r}\Bigg|_{\alpha=-z_l} \qquad C_n^k = \frac{n!}{k!(n-k)!}$$

$v = 0$ for $p(l) = 0$.

Here $K_{\pm}(u) = K_s^{\pm}(u)$ is the result of factorization [19] of function $K_s(u)$, i.e. the representation of the latter in the form of a product of functions which are regular in the upper and lower planes

$$K_s(u) = K_s^+(u) K_s^-(u), \qquad K_s^+(-u) = K_s^-(u)$$

Apparently,

$$K_s^+(u) = \Gamma(g/\beta - iu/\beta)\,\Gamma^{-1}(b/\beta - iu/\beta)\left[1 + \sum_{k=1}^{s} c_k \varphi_k(u)\right]_+$$

Lemma 3.3. The operator of imbedding of $S(\sigma)$ into $s(\lambda)$, $\lambda < \sigma$ is completely continuous.

This Lemma is easy to prove if we take into account that $s(\lambda)$ is a space with a base [20].

3°. In order to prove the solvability of system (3.8), let us examine Eq. (1.1) with a right side of the form

$$f_{2m-1}(x) = \int_{a_{2m-1}}^{\infty} k(x-\xi)\,\sigma_{2m-1}(\xi)\,d\xi + \int_{-\infty}^{a_{2m}} k(x-\xi)\sigma_{2m}(\xi)\,d\xi \qquad (3.9)$$

$$a_{2m-1} \leqslant x \leqslant a_{2m}$$

Here

$$\sigma_{2m-1}(\xi) = \sum_{l=1}^{\infty}\sum_{v=0}^{p(l)} V_l(2m-1, v)\,(\xi - a_{2k-1})^v \exp iz_l(\xi - a_{2m-1})$$

$$\sigma_{2m}(\xi) = \sum_{l=1}^{\infty}\sum_{v=0}^{p(l)} V_l(2m, v)\,(a_{2m} - \xi)^v \exp iz_l(a_{2m} - \xi) \qquad (3.10)$$

With regard to sequences $V(k) = \{V_l(k, v)\}$ it is assumed that they are arbitrary elements from $s(\sigma)$

$$1-2\gamma < \sigma < 1, \quad \gamma \leqslant 0.5; \qquad 0 < \sigma < 1, \quad 0.5 < \gamma$$

As a result of these assumptions functions $\sigma_k(\xi)$ belong to the class of uniqueness indicated in Theorem 2.2.

As in Sect. 3. 2°, we reduce the integral equation (1.1) with the right side (3.9) to an infinite system of linear algebraic equations. The solution of the integral equation will be sought in the form (3.7) for $F(\eta) \equiv 0$. As a result we obtain the infinite system (3.8) in the right side of which are the following elements:

$$L_k = AV(2k-1), \qquad G_k = AV(2k) \qquad (3.11)$$

We operate from the left with the matrix A^{-1} on the system (3.8) which has the right side as in (3.11). As a result we arrive at an equation of the second kind which symbolically can be represented in the form

$$X + \mathbf{U}X = V \qquad (3.12)$$

Here V is an element of a space of a $2N$-dimensional infinite sequence. The lth component of this space is $V_l(1, s)$, $V_l(2, s)$, ..., $V_l(2N, s)$. The element X also belongs to this space.

It is completely clear that the introduction of this space only simplifies the form of notation for the infinite system. The essence of the infinite system (3.12), however, consists in the fact that the operator $\mathbf{U}$ by virtue of Lemmas 3.2 and 3.3 is completely continuous in any $s(\lambda)$, $\lambda < 1 - \gamma$.

This circumstance turns out to be essential in the proof of solvability of system (3.12) and together with it also of integral equations (1.1).

In fact, in its structure the system (3.12) is equivalent to the integral equation (1.1) with the right side (3.9). Since Eq. (1.1) cannot have more than one solution, then by virtue of earlier indicated properties of Dirichlet series utilized here, the infinite system (3.12) also cannot have more than one solution in any $s(\lambda), \lambda > 0$. However, the right side of system (3.12) is an arbitrary element of Banach space. Thus, Eq. (3.12) is an equation of the second kind in Banach space with a completely continuous operator. This equation also does not permit more than one solution for any arbitrary right side. This means [21] that the index of the operator and both defective numbers are zeros, i.e. Eq. (3.12) is uniquely solvable for any right side

$$V \in s(\lambda), \qquad \lambda < 1 - \gamma$$

In order to construct a solution of system (3.12) it can be reduced [22, 11] to some finite system of linear equations. It was established above that the determinant for this system is not equal to zero.

It turns out that as a final result the solution X of system (3.12) can be represented in the form

$$X = (I + \mathbf{U})^{-1} V, \qquad \|X\|_{s(\lambda)} \leqslant \|(I + \mathbf{U})^{-1}\| \cdot \|V\|_{s(\lambda)} \tag{3.13}$$

The obtained results will be applied to system (3.8).

Lemma 3.4. Let $f(x) \in E$. Then the solution of the system (3.8) belongs to $S(1 - \gamma)$ and the following estimate is valid:

$$\|X\|_{S(1-\gamma)} \leqslant \Lambda \|f\|_E, \qquad \Lambda = \mathrm{const} \tag{3.14}$$

The proof follows from the fact that the free term of the system of the second kind belongs to the space $S(1 - \gamma)$, and the resolvent $(I + \mathbf{U})^{-1}$ is continuous in this space.

Taking into consideration the result of Lemma 3.4 and applying it to the series (3.7), we prove the following theorem.

Theorem 3.1. Let $f(x) \in E$. Then for $K(u) = K_s(u)$ the following estimate is valid

$$\|q\|_{C(\gamma)} \leqslant \|\mathbf{K}_s^{-1}\| \|f\|_E \tag{3.15}$$

The proof of this theorem is omitted.

As a result of Lemma 3.4 combined with the Euler-Maclaurin formula for the series (3.7) it is sufficiently simple to obtain (3.15).

We point out that $\|\mathbf{K}_s^{-1}\|$ can be computed in the case of large a for $N = 1$ utilizing formulas of [23] which remain valid for $0 < \gamma < 1$.

4. Now let us turn to the proof of solvability of Eq. (1.1) with a general kernel. We examine Eq. (2.3) in which $K(u)$ and $M(u)$ coincide with $K_s(u)$ and $M_s(u)$, respectively. For $\lambda = 1$ Eq. (2.3) is apparently identical to (1.1).

By virtue of property (3.2) it is possible to select such an $s > 0$ that the condition of Theorem 2.2 is satisfied. This means that Eq. (2.3) for all λ from the circle (2.6) cannot have more than one solution (and may not have any at all!) in L_p which is given by Theorem 2.2.

We note that $C(\tau)$, $\tau < 2\gamma$, $\gamma < 0.5$; $\tau < 1$, $\gamma \geqslant 0.5$ is imbedded in the indicated L_p.

We will show that Eq. (2.3) has a solution in $C(\gamma)$. Let $f(x) \in E$. Then by virtue of Lemma 3.4 and taking into account the estimate (2.2) for selected $s > 0$, it is possible to represent Eq. (2.3) in a symbolic manner in the following form:

$$q + \lambda \mathbf{K}_s^{-1}\mathbf{M}_s q = \mathbf{K}_s^{-1} f \tag{4.1}$$

Here $q(x)$ is a vector-function determined on finite segments $[a_{2k-1}, a_{2k}]$. The operator $\mathbf{K}_s^{-1}$ is constructed on the basis of solution (3.13) of an infinite system of linear equations.

The following lemma turned out to be the most complicated in this work.

Lemma 4.1. The operator $\mathbf{K}_s^{-1}\mathbf{M}_s$ is completely continuous in

$C(\tau)$, $\gamma \leqslant \tau < \varkappa^\circ$, $\varkappa^\circ = \inf(\delta, 2\gamma)$, $\gamma < 0.5$; $\varkappa^\circ = \inf(\delta, 1)$, $\gamma \geqslant 0.5$.

Proof. The operator $\mathbf{K}_s^{-1}$ is constructed through the application of a certain iteration process to the infinite system of equations and by subsequent solution of a uniquely invertible finite system [11]. In just this manner it is possible to analyze successfully the operator $\mathbf{K}_s^{-1}$. The operator $\mathbf{K}_s^{-1}\mathbf{M}_s$ represents a superposition of operators which are analogous to the ones presented below

$$\mathbf{P}q \equiv \int_{-\infty}^{\infty} \frac{M_s(u)}{K_s(u)} Q(u)\, e^{iux} du$$

$$\mathbf{R}q \equiv \sum_{k=1}^{N} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{-it(a_{2k}-x)} + e^{-it(x-a_{2k-1})}}{K_s^{+}(t)} \int_{-\infty}^{\infty} \frac{M_s(u)\, Q(u)\, du dt}{K_s^{+}(u)(u-t)} \tag{4.2}$$

Here $\varepsilon > 0$ does not exceed the half-width of the band of regularity of the function $K_s(u)$.

The complete continuity of operators $\mathbf{P}$ and $\mathbf{R}$ must be verified on segments $[a_{2k-1}, a_{2k}]$, $k = 1, 2, \ldots, N$ in the space $C(\tau)$. This indicates that sets of functions $\mathbf{P}q$ and $\mathbf{R}q$ for $\|q\|_{C(\tau)} < B$ with the weight $(a_{2k}-x)^\tau (x-a_{2k-1})^\tau$ must be compact on each of segments $[a_{2k-1}, a_{2k}]$ in the metric of the space C.

The function $Q(u)$ which appears in (4.2) is given by the relationship (1.4).

An estimate of the form

$$K_s^{+}(z) = cz^{-\gamma}[1 + O(z^{-1})], \quad |z| \to \infty, \quad |\arg z + \pi/2| > 0$$

makes it easier to check the indicated properties of the operator R.

Lemma 4.1 makes it possible to prove the solvability of Eq. (4.1), which, as was mentioned at the beginning of this Section, cannot have more than one solution (but may have none at all) for

$$|\lambda| < \varkappa^{-1} = 1 + \theta, \quad \theta > 0 \tag{4.3}$$

Let function f in Eq. (2.3) be given by the relationship

$$f = \mathbf{K}_s\varphi, \quad \varphi(x) \in C(\tau), \quad \gamma \leqslant \tau < \varkappa^\circ \tag{4.4}$$

The selected function $\varphi(x)$ belongs to L_p indicated in Theorem 2.2.

Such a choice of function f permits to represent Eq. (4.1) in the form

$$q + \lambda \mathbf{K}_s^{-1}\mathbf{M}_s q = \varphi \tag{4.5}$$

Equation (4.5) is an equation of the second kind with completely continuous operator in the Banach space. It can have no more than one solution for any right side from this space for any λ from a circle (4.3). Using the same reasoning as in Sect. 3.3°, we conclude that (4.5) is uniquely solvable for all $|\lambda| < \varkappa^{-1}$, this includes also $\lambda = 1$.

The solution of (4.5) can also be obtained by the method of successive approximations.

The process converges for all $|\lambda| < \varkappa^{-1}$. This follows from the fact that the spectral radius ρ of the operator (4.5) (radius of a circle which does not contain points of the spectrum of the operator), is no less than $\varkappa^{-1}$.

Thus

$$\|q\|_{C(\gamma)} \leqslant C\|q\|_{C(\gamma)}, \qquad C = \left\| \sum_{m=0}^{\infty} \lambda^m (\mathbf{K}_s^{-1}\mathbf{M}_s)^m \right\| \tag{4.6}$$

Returning now to Eq. (2.3) we conclude that it is uniquely solvable in $C(\gamma)$ for all f such that

$$\mathbf{K}_s^{-1} f \in C(\gamma) \tag{4.7}$$

It is evident from Theorem 3.1 that condition (4.7) is valid for $f \in E$.

Furthermore

$$\|q\|_{C(\gamma)} \leqslant \|\mathbf{K}^{-1}\| \cdot \|f\|_E, \qquad \|\mathbf{K}^{-1}\| = C\|\mathbf{K}_s^{-1}\| \tag{4.8}$$

In applications the most frequently encountered case [1] is $\gamma = 0.5$. It can be shown that in this particular case for relationship (4.7) to be valid it is sufficient that

$$f \in C_1^{\lambda} \qquad (\lambda > 0.5) \tag{4.9}$$

For these right parts the following estimate is applicable

$$\|q\|_{C(0.5)} \leqslant M\|f\|_{C_1^\lambda} \tag{4.10}$$

Inequalities (4.8), (4.10) are relationships of correctness for the integral equation (1.1). They indicate that a small change in the right side of Eq. (1.1) in E and (C_1^{λ}) leads to a small change of the solution in $C(\gamma)$, $(C(0.5))$.

5. The obtained results are applied to a proof of the method of approximate factorization. Let there be two integral equations of the form

$$\mathbf{K}_1 q_1 = 2\pi f, \qquad \mathbf{K}_2 q_2 = 2\pi f$$

It is assumed that their kernels have the form (1.2) and that in this case the Fourier trasforms $K_1(u)$ and $K_2(u)$, respectively, satisfy conditions (1)–(3) of Sect. 1, and also that $f \in E$.

Theorem 5.1. Let the quantity $\varepsilon > 0$

$$\varepsilon = \max |K_1(u) - K_2(u)| K_1^{-1}(u)(1 + |u|)^{\alpha} \quad (u \in [-\infty, \infty])$$

$$\alpha > 1 - \gamma, \quad \gamma < 0.5; \quad \alpha > \gamma, \quad 0.5 \leqslant \gamma$$

satisfy the condition

$$\varepsilon < \|\mathbf{K}_1^{-1}\|^{-1} L^{-1}$$

Then the following estimate is valid

$$\|q_2 - q_1\|_{C(\gamma)} \leqslant \varepsilon L \|\mathbf{K}_1^{-1}\| (1 - \varepsilon L \|\mathbf{K}_1^{-1}\|)^{-1} \|q_1\|_{C(\gamma)}$$

$$L = \frac{2B(1 - p\gamma, 1 - p\gamma)}{\alpha p - 1} \sum_{k=1}^{N} (a_{2k} - a_{2k-1})^{\varkappa} \qquad \left(\varkappa = \frac{1 - 2p\gamma}{p}\right)$$

$$\alpha^{-1} < p < \gamma^{-1}, \quad 0.5 \leqslant \gamma; \qquad p = 2, \quad \gamma < 0.5$$

Here $B(x, y)$ is Euler's beta function. The quantity $\|K_1^{-1}\|$ is given by the relationship (4.8). The theorem is proved by a method well known in the theory of perturbations. This method is based on successive approximations. It is necessary only to keep in mind that the operator $\mathbf{K}_1 - \mathbf{K}_2$ acts continuously from $C(\gamma)$ into E and its norm does not exceed the quantity εL.

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